

Stokes' Problems in Kinetic Theory

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Stokes' problems, concerning the steady flow of a viscous fluid past a sphere and a circular cylinder, have been solved using the linearized Boltzmann equation. The solutions obtained are well behaved at infinity; Stokes' and Whitehead's paradoxes do not occur. For the sphere the well-known drag formula $P = 6\pi\eta r\eta\rho(1 + \frac{3}{8}R \dots)$ is found. The results for the drag on the cylinder are in better agreement with experimental values than the classical formula of Lamb.

I. INTRODUCTION

Stokes' problem, concerning the steady flow of a viscous fluid past a sphere, is a peculiar case in fluid dynamics because one must be careful in the linearization of the hydrodynamical equations. The question arises as to whether similar difficulties affect the linearization of the Boltzmann equation. In the framework of the Chapman-Enskog theory, it has been concluded¹ that the kinetic treatment does not avoid these difficulties and in particular, that the solutions are badly behaved at infinity (the so-called Stokes' and Whitehead's paradoxes). These conclusions are somewhat confusing. It was already pointed out by Grad² that linear and non-linear results are related quite differently in the fluid and kinetic versions. The fluid equations are obtained from the Boltzmann equation by operating only on the streaming term, which is taken into account exactly in the linearized Boltzmann equation. Indeed, if one takes the *linearized* Boltzmann equation as it stands, but defines the hydrodynamical quantities in the exact manner as given from the full Boltzmann distribution function (without linearizing), applying the well-known moment procedure, one gets the *nonlinear* hydrodynamical equations. Hence, the defects arising from linearization of the hydrodynamical equations cannot be contained in the linearized Boltzmann equation. In this paper this will be illustrated by the explicit solution of Stokes' problem with the linearized Boltzmann equation in two and three dimensions.

In order to treat boundary value problems for the Boltzmann equation, the Chapman-Enskog procedure is not very appropriate because it is quite difficult to treat the boundary conditions consistently. A more powerful tool is the method of normal solutions, previously developed.^{3,4} This method offers considerable advantages: it is a quite general procedure that works in various geometrical and

hydrodynamical situations, the computational work consists of nothing but the calculation of Fourier integrals, and higher approximations, besides the Navier-Stokes level, can be carried out in a straightforward manner. The only disadvantage is that the method is restricted to the hydrodynamical region; the Knudsen region is excluded.

This method is described briefly in the next section. In the third section it is applied to Stokes' problem for the sphere. The solution obtained decreases more rapidly to the uniform stream at infinity than Stokes' solution, but reduces to it in the limit of small Reynolds numbers R . In this case we get the well-known expression for the drag on the sphere

$$P = 6\pi\eta_0 r\eta\rho(1 + \frac{3}{8}R + \dots).$$

Stokes' problem for the cylinder is solved in the same way in the next section. The solution is again well behaved at infinity; Stokes' paradox does not occur. For the drag on the cylinder we get

$$P = \eta_0^2 r \rho 2\pi \left(R \int_0^\infty J_0^2(kR) [(k^2 + 1)^{1/2} - k] dk \right)^{-1},$$

where $J_0(z)$ is the Bessel function of the first kind. For small Reynolds numbers R this value agrees with the classical result of Lamb, but it is much better at higher R if compared with the experimental values of Tritton.⁵

The last section is concerned with a discussion of the results, especially in view of the problem of the boundary condition on the surface of the body.

II. THE NORMAL SOLUTION METHOD

We summarize the main relations of the method of normal solutions which are necessary to treat steady-state problems for the linearized Boltzmann equation. For details of this method we refer to previous papers.^{3,4}

Let us start with the linearized Boltzmann equation

$$\frac{\partial f}{\partial t} = -\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - If \stackrel{\text{def}}{=} Bf. \quad (1)$$

The function $f(\mathbf{x}, \mathbf{v}, t)$ determines the Boltzmann distribution function $F(\mathbf{x}, \mathbf{v}, t) = \varphi_0(\mathbf{v})(1 + f)$, where $\varphi_0(\mathbf{v})$ is a space-independent Maxwell distribution with mean density $n = 1$ and mean velocity $\mathbf{v}_0 = 0$. We choose dimensionless quantities. By I we denote the linearized collision operator

$$If = \int [f(\mathbf{v}) + f(\mathbf{v}_1) - f(\mathbf{v}') - f(\mathbf{v}'_1)] \cdot |\mathbf{v} - \mathbf{v}_1| \sigma \varphi_0(\mathbf{v}_1) d^2\Omega d^3v_1. \quad (2)$$

The Boltzmann operator denoted by B always refers to an infinite system without boundaries covering the whole of $\mathbf{R}^n$ ($n \leq 3$). This operator defined on an appropriate domain is a dissipative operator in Hilbert space

$$\mathcal{H} = L^2(\mathbf{R}^n) \otimes L^2_{\varphi_0}(\mathbf{R}^n)$$

which consists of functions $f(\mathbf{x}, \mathbf{v})$ with the scalar product given by

$$\begin{aligned} (f, g)_{x,v} &= \int \bar{f}(\mathbf{x}, \mathbf{v}) g(\mathbf{x}, \mathbf{v}) \varphi_0(\mathbf{v}) d^n v d^n x \\ &= \int (f, g)_v d^n x. \end{aligned}$$

The Boltzmann operator B can be handled by means of the normal solutions. The latter are eigenfunctions of the Fourier transformed operator

$$\hat{B}_k = -i\mathbf{k} \cdot \mathbf{v} - I \quad (3)$$

consider as an operator in $L^2_{\varphi_0}(\mathbf{R}^n)$ with $\mathbf{k}$ fixed

$$\hat{B}_k g_i(\mathbf{k}, \mathbf{v}) = -i\omega_i(\mathbf{k}) g_i(\mathbf{k}, \mathbf{v}). \quad (4)$$

The eigenvalues $-i\omega_i$ and the eigenfunctions g_i can be calculated by analytical perturbation theory.⁴ The result up to first and second order in k , respectively, which is the Navier-Stokes approximation, is as follows ($n = 3$):

$$\begin{aligned} -i\omega_{1,2}(k) &= -\eta k^2, \\ -i\omega_3(k) &= -\frac{2}{5}\lambda k^2, \\ -i\omega_{4,5}(k) &= \mp i\kappa k - \Gamma k^2, \end{aligned} \quad (5)$$

$$\begin{aligned} g_1(\mathbf{k}, \mathbf{v}) &= \mathbf{v} \cdot \mathbf{e}^1 - i \sum_{ij} T_{ij} k_i e_j^1, \\ g_2(\mathbf{k}, \mathbf{v}) &= \mathbf{v} \cdot \mathbf{e}^2 - i \sum_{ij} T_{ij} k_i e_j^2, \\ g_3(\mathbf{k}, \mathbf{v}) &= \left(\frac{2}{5}\right)^{1/2} \left(u - \frac{5}{2} + i\frac{2}{5}\lambda \mathbf{v} \cdot \mathbf{k} - i\mathbf{S} \cdot \mathbf{k}\right), \end{aligned} \quad (6)$$

$$\begin{aligned} g_{4,5}(\mathbf{k}, \mathbf{v}) &= \left(\frac{2}{15}\right)^{1/2} (u - i\eta_1 \mathbf{v} \cdot \mathbf{k} - i\mathbf{S} \cdot \mathbf{k}) \\ &\pm 2^{-1/2} (\mathbf{v} \cdot \mathbf{e}^3 + i\frac{2}{5}\lambda k + i\lambda_1 k u - i \sum_{ij} T_{ij} k_i e_j^3), \end{aligned}$$

$$k = |\mathbf{k}|, \quad u = \frac{1}{2}v^2, \quad c = \left(\frac{5}{3}\right)^{1/2}, \quad \Gamma = \frac{2}{3}(\eta + \frac{1}{5}\lambda),$$

$$\eta_1 = \frac{1}{2}\eta - \frac{1}{15}\lambda, \quad \lambda_1 = \frac{2}{15}\eta - \frac{1}{75}\lambda, \quad \mathbf{e}^3 = \frac{\mathbf{k}}{k}.$$

$\mathbf{e}^1, \mathbf{e}^2$, and $\mathbf{e}^3$ are unit vectors with $\mathbf{e}^3$ parallel and $\mathbf{e}^1, \mathbf{e}^2$ perpendicular to $\mathbf{k}$. The quantities $\mathbf{S}$ and T_{ij} are the solutions of the integral equations

$$IS = (u - \frac{5}{2})\mathbf{v}, \quad IT_{ij} = v_i v_j - \frac{1}{3}\delta_{ij} v^2$$

with

$$(S_j, w_\ell)_v = 0, \quad (T_{ij}, w_\ell) = 0, \quad \ell = 1, \dots, 5,$$

where $w_\ell = 1, \mathbf{v}, \frac{1}{2}v^2$ are the summational invariants. The thermal conductivity λ and the viscosity η are determined by

$$((u - \frac{5}{2})v_i, S_j)_v = \lambda \delta_{ij},$$

$$(v_i v_j - \frac{1}{3}\delta_{ij} v^2, T_{\ell m})_v = \eta (\delta_{i\ell} \delta_{jm} + \delta_{im} \delta_{j\ell} - \frac{2}{3}\delta_{ij} \delta_{\ell m}).$$

Using these results we are able to calculate functions of the operator B ; for instance, we have the following important formula for the inverse B^{-1} :

$$\begin{aligned} (B^{-1}f)(\mathbf{x}, \mathbf{v}) &= \frac{1}{(2\pi)^3} \sum_{i=1}^5 \int d^3k \frac{f_i(\mathbf{k})}{-i\omega_i(k)} g_i(\mathbf{k}, \mathbf{v}) \exp(i\mathbf{k} \cdot \mathbf{x}). \end{aligned} \quad (7)$$

The coefficients $f_i(\mathbf{k})$ in (2.7) are given by

$$f_i(\mathbf{k}) = \int (h_i, f)_v \exp(-i\mathbf{k} \cdot \mathbf{x}) d^3x, \quad (8)$$

where the h_i are biorthogonal to the g_i : $(g_j, h_i)_v = \delta_{ji}$. In the Navier-Stokes approximation (6) the g_i and h_i are equal. The expression (7) is the so-called hydrodynamic part of the general solution; microscopic relaxation is neglected. It is correct to take only this hydrodynamic part if $\mathbf{x}$ is out of the kinetic boundary layers. A careful investigation of the hydrodynamic approximation is given in Ref. 4.

Now, we turn to boundary value problems. The essential quality of the boundaries is that they are sources of energy or momentum. Therefore, a steady-state situation will be described by the time-independent Boltzmann equation

$$-\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - If = 0$$

for $\mathbf{x}$ in the region V covered by the fluid, but on the boundaries V' we may write

$$-\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - If = -h(\mathbf{x}, \mathbf{v}) \quad \mathbf{x} \in V'. \quad (9)$$

The source function $h(\mathbf{x}, \mathbf{v})$ represents the sources of energy or momentum (or both) and is zero for $\mathbf{x} \in V$. If the source function is known, the solution of the steady-state problem is given by

$$f = -B^{-1}h. \quad (10)$$

We previously found that in various interesting cases the source function $h(\mathbf{x}, \mathbf{v})$ factors according to

$$h(\mathbf{x}, \mathbf{v}) = a(\mathbf{x})h_1(\mathbf{v}) \delta_{V'}(\mathbf{x}), \quad (11)$$

where $h_1(\mathbf{v})$ is equal to $u - \frac{3}{2}$ for energy production or equal to $\mathbf{v}$ for momentum production and $\delta_{V'}$ is a δ function concentrated on the boundary V' . For illustration let us give some examples: For heat flow between two parallel plates of distance $2d$ the source function is⁴

$$h(\mathbf{x}, \mathbf{v}) = a(u - \frac{3}{2})[\delta(x_1 - d) - \delta(x_1 + d)],$$

for Couette flow one must take

$$h(\mathbf{x}, \mathbf{v}) = av_2[\delta(x_1 - d) - \delta(x_1 + d)].$$

The constant a is proportional to the difference of the temperatures or velocities at the two walls, or at the same time proportional to the energy or momentum production rate. In any case the source function has to be justified by calculating the steady-state solution (10) and verifying that it gives the right boundary values apart from boundary layer effects.

Now let us consider Stokes' problem of a fluid moving in the three direction and a sphere of radius r taken at rest at the origin. The source function corresponding to this situation is

$$h(\mathbf{x}, \mathbf{v}) = av_3 \delta(|\mathbf{x}| - r). \quad (12)$$

In contrast to the examples given above it is not clear that the x dependence of $a(\mathbf{x})$ in Eq. (11) is simply the constant a again, because the sources of momentum could be distributed nonuniformly over the surface. Nevertheless, as will be seen, the source function (12) is correct at least for small Reynolds numbers. We return to this point in the last section.

Since the inhomogeneous term in Eq. (9) must be time independent (12), we calculate in the rest frame with respect to the sphere. Therefore, all quantities must be transformed into this system.

We make a Galilei transformation

$$\mathbf{x}' = \mathbf{x} + \mathbf{v}_0 t, \quad \mathbf{v}' = \mathbf{v} + \mathbf{v}_0, \quad t' = t. \quad (13)$$

The Boltzmann equation (1) is Galilei invariant

$$\frac{\partial f'}{\partial t'} = -\mathbf{v}' \cdot \frac{\partial f'}{\partial \mathbf{x}'} - I'f',$$

where

$$f'(\mathbf{x}; \mathbf{v}', t') = f(\mathbf{x}' - \mathbf{v}_0 t', \mathbf{v}' - \mathbf{v}_0, t')$$

and I' is the linearized collision operator with a Maxwellian with mean velocity $\mathbf{v}_0$

$$\varphi'_0(\mathbf{v}') = (2\pi)^{-3/2} \exp[-\frac{1}{2}(\mathbf{v}' - \mathbf{v}_0)^2].$$

We have

$$(I'f')(\mathbf{x}', \mathbf{v}', t') = (If)(\mathbf{x}, \mathbf{v}, t).$$

Choosing $t = t' = 0$ we obtain for any function $g(\mathbf{k}, \mathbf{v})$ in the domain of I by Fourier transformation with respect to $\mathbf{x}$

$$(I'g')(\mathbf{k}, \mathbf{v}') = (Ig)(\mathbf{k}, \mathbf{v}), \quad (14)$$

where

$$g'(\mathbf{k}, \mathbf{v}') = g(\mathbf{k}, \mathbf{v}' - \mathbf{v}_0) \quad (15)$$

which can be found directly from Eq. (2) also. From Eqs. (14) and (15) we get the eigenvalues and eigenfunctions or $\hat{B}'_k$

$$\begin{aligned} \hat{B}'_k g'_i &= -i\mathbf{k} \cdot \mathbf{v}' g'_i - I' g'_i = -i\mathbf{k} \cdot \mathbf{v} g'_i - i\mathbf{k} \cdot \mathbf{v}_0 g'_i - I g_i \\ &= -i\mathbf{k} \cdot \mathbf{v}_0 g_i - i\omega_i(k) g_i, \end{aligned}$$

that is,

$$g'_i(\mathbf{k}, \mathbf{v}') = g_i(\mathbf{k}, \mathbf{v}' - \mathbf{v}_0), \quad (16)$$

$$-i\omega'_i(\mathbf{k}) = -i\mathbf{k} \cdot \mathbf{v}_0 - i\omega_i(k). \quad (17)$$

Using these transformed eigenvalues and normal solutions in Eqs. (7) and (8) we can calculate the inverse B'^{-1} which gives the solution of the steady-state problem. The scalar products with respect to $\mathbf{v}$ are invariant

$$(f'(\mathbf{v}'), g'(\mathbf{v}'))_{\mathbf{v}'} = (f(\mathbf{v}), g(\mathbf{v}))_{\mathbf{v}}.$$

Hence, it is possible to calculate with the previous normal solutions (6) recalling that $\mathbf{v} = \mathbf{v}' - \mathbf{v}_0$ is now the velocity relative to the frame moving with the fluid.

III. STOKES' PROBLEM FOR THE SPHERE

We choose spherical coordinates

$$\mathbf{x} = (x_1, x_2, x_3) = x(\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta),$$

$$\mathbf{k} = (k_1, k_2, k_3) = k(\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha)$$

$$= k\mathbf{e}_3,$$

$$\mathbf{e}^1 = (-\cos \alpha \cos \beta, -\cos \alpha \sin \beta, \sin \alpha),$$

$$\mathbf{e}^2 = \mathbf{e}^3 \times \mathbf{e}^1 = (\sin \beta, -\cos \beta, 0).$$

Then, we have for the normal solutions (6)

$$\begin{aligned}
g_1 &= -v_1 \cos \alpha \cos \beta - v_2 \cos \alpha \sin \beta + v_3 \sin \alpha \\
&\quad - ik[-T_{11} \sin \alpha \cos \alpha \cos^2 \beta \\
&\quad - T_{12} \sin \alpha \cos \alpha \sin \beta \cos \beta + T_{13} \sin^2 \alpha \cos \beta \\
&\quad - T_{21} \sin \alpha \cos \alpha \sin \beta \cos \beta \\
&\quad - T_{22} \sin \alpha \cos \alpha \sin^2 \beta + T_{23} \sin^2 \alpha \sin \beta \\
&\quad - T_{31} \cos^2 \alpha \cos \beta \\
&\quad - T_{32} \cos^2 \alpha \sin \beta + T_{33} \sin \alpha \cos \alpha], \\
g_2 &= v_1 \sin \beta - v_2 \cos \beta - ik[\dots], \\
g_3 &= (\frac{2}{5})^{1/2}[u - \frac{5}{2} + i\frac{2}{5}\lambda k(v_1 \sin \alpha \cos \beta \\
&\quad + v_2 \sin \alpha \sin \beta + v_3 \cos \alpha) - i\mathbf{S} \cdot \mathbf{k}], \\
g_{4,5} &= (\frac{2}{15})^{1/2}[u - i\eta_1 k(v_1 \sin \alpha \cos \beta \\
&\quad + v_2 \sin \alpha \sin \beta + v_3 \cos \alpha) - i\mathbf{S} \cdot \mathbf{k}] \\
&\quad \pm 2^{-1/2}[v_1 \sin \alpha \cos \beta + v_2 \sin \alpha \sin \beta \\
&\quad + v_3 \cos \alpha + ik(\dots)].
\end{aligned} \tag{18}$$

The eigenvalues (17) are

$$\begin{aligned}
-i\omega'_{1,2} &= -ikv_0 \cos \alpha - \eta k^2, \\
-i\omega'_3 &= ikv_0 \cos \alpha - \frac{2}{5}\lambda k^2, \\
-i\omega'_{4,5} &= -ikv_0 \cos \alpha \mp ick - \Gamma k^2.
\end{aligned}$$

As source function we take (12)

$$h(\mathbf{x}, \mathbf{v}) = av_3 \delta(x - r). \tag{19}$$

It is very simple to calculate the momentum flux generated by this source which, in the rest frame of the sphere, gives the drag P on the sphere. Multiplying the inhomogeneous Boltzmann equation (9) by v_3 and φ_0 , integrating with respect to $\mathbf{v}$ and $\mathbf{x}$ and using Gauss' theorem we get the momentum flux through an arbitrary closed surface S around the sphere

$$-\int \tau'_{i3} d\sigma_i = P = -4\pi ar^2 \tag{20}$$

which is equal in the two coordinate systems.

Inserting the expressions (19) and (18) into Eq. (8) we observe that the terms with $j = 1, 3, 4, 5$ contribute to the total solution (10). As we shall see, the first mode $j = 1$ gives the velocity field and the stress tensor, the fourth and fifth modes $j = 4, 5$ give the pressure and boundary layers to the velocity. The modes $j = 3, 4, 5$ contain the variation of temperature and density, the heat flow, and various boundary layers.

Now, we calculate the first mode contribution $f^1(\mathbf{x}, \mathbf{v})$. Since

$$(g_1, h)_v = a \sin \alpha \delta(x - r)$$

we get

$$\begin{aligned}
f_1(\mathbf{k}) &= a \sin \alpha \int d^3x \delta(x - r) \exp(-i\mathbf{k}\mathbf{x}) \\
&= 4\pi ar^2 \sin \alpha \frac{\sin kr}{kr}
\end{aligned}$$

and

$$\begin{aligned}
f^1(\mathbf{x}, \mathbf{v}) &= -\frac{4\pi ar^2}{(2\pi)^3} \int d^3k \frac{\sin \alpha}{-ikv_0 \cos \alpha - \eta k^2} \frac{\sin kr}{kr} \\
&\quad \cdot (-v_1 \cos \alpha \cos \beta - v_2 \cos \alpha \sin \beta \\
&\quad + v_3 \sin \alpha - ik \dots) \exp(i\mathbf{k}\mathbf{x}). \tag{21}
\end{aligned}$$

It is convenient to discuss the behavior at infinity $x \rightarrow \infty$ in this Fourier integral. The term which is most singular at $\mathbf{k} = 0$ is the term proportional to v_3 , say $A_3(x)$. One easily estimates that

$$\frac{\sin^2 \alpha}{ikv_0 \cos \alpha + \eta k^2} \frac{\sin kr}{kr} \in L^p(\mathbf{R}^3) \quad \text{with} \quad \frac{3}{2} < p < 2.$$

Applying the Hausdorff-Yang theorem it follows that

$$A_3(x) \in L^{2+\epsilon}(\mathbf{R}^3), \quad \epsilon > 0.$$

In the Stokes' solution this term is proportional to $1/x$ which is $L^{3+\epsilon}(\mathbf{R}^3)$ only. Hence, it is already clear that our solution (21) behaves better at infinity than Stokes' solution; Whitehead's paradox does not occur here.

Computing the Fourier integral (21) in spherical coordinates we obtain

$$\begin{aligned}
A_3(\mathbf{x}) &= \frac{2ar}{(2\pi)^2} \int \frac{\sin^3 \alpha}{iv_0 \cos \alpha + \eta k} \sin kr \\
&\quad \cdot \exp[ikx(\sin \alpha \sin \vartheta \cos(\beta - \varphi) \\
&\quad + \cos \alpha \cos \vartheta)] d\beta d\alpha dk \\
&= \frac{2ar}{2\pi} \int \frac{\sin^3 \alpha}{iv_0 \cos \alpha + \eta k} \sin kr J_0(kx \sin \alpha \sin \vartheta) \\
&\quad \cdot \exp(ikx \cos \alpha \cos \vartheta) d\alpha dk.
\end{aligned}$$

We expand into Legendre polynomials⁶

$$\begin{aligned}
&J_0(kx \sin \alpha \sin \vartheta) \exp(ikx \cos \alpha \cos \vartheta) \\
&= \left(\frac{2\pi}{kx}\right)^{1/2} \sum_{\ell=0}^{\infty} i^{\ell} \left(\ell + \frac{1}{2}\right) J_{\ell+1/2}(kx) P_{\ell}(\cos \alpha) P_{\ell}(\cos \vartheta),
\end{aligned}$$

then we have

$$A_3(\mathbf{x}) = \sum_{\ell} A'_{\ell}(x) P_{\ell}(\cos \vartheta). \tag{22}$$

If the velocity field is calculated, we get the stress tensor from the Navier-Stokes relations, because the terms with T_{ij} in Eq. (18) are just derivatives of the corresponding terms with v_i . Then, the velocity terms contributing to the drag are those with $\ell = 0, 2$ in Eq. (22). Integrating with respect to α for A_3^0 we obtain

$$A_3^0(x) = \frac{ar}{\pi} \frac{2\eta}{v_0^2} \frac{1}{x} \int_0^\infty dk \frac{\sin kr}{k} \cdot \sin kx k \left[-1 + \left(\frac{v_0}{\eta k} + \frac{\eta k}{v_0} \right) \operatorname{arctg} \frac{v_0}{\eta k} \right].$$

This integral can be carried out by contour integration in the complex k plane. The result is

$$A_3^0(x) = -\frac{ar}{(2|v_0|x)} \left\{ E_1(x_2) + \log x_2 - E_1(x_1) - \log x_1 + \frac{1}{x_2^2} - \frac{1}{x_1^2} + \left(\frac{1}{x_1} + \frac{1}{x_1^2} \right) \cdot \exp(-x_1) - \left(\frac{1}{x_2} + \frac{1}{x_2^2} \right) \exp(-x_2) \right\},$$

where

$$x_1 = \frac{v_0(x+r)}{\eta}, \quad x_2 = \frac{v_0(x-r)}{\eta}, \quad (23)$$

$$E_1(z) = \int_z^\infty dt \frac{\exp(-t)}{t}.$$

In the limit $v_0/\eta \rightarrow 0$ ($v_0 > 0$) this gives

$$A_3^0(x) = \frac{ar}{v_0 x} \left(\frac{2}{3} R - \frac{1}{4} R \frac{v_0 x}{\eta} \right), \quad (24)$$

$$\begin{aligned} A_3^2(x) &= -\frac{ar}{\pi} \frac{15\eta}{v_0^2} \frac{1}{x} \int_0^\infty dk \sin kr \left[\left(\frac{3}{k^2 x^2} - 1 \right) \sin kx - \frac{3}{kx} \cos kx \right] \left(1 + \frac{\eta^2}{v_0^2} k^2 \right) \left[1 - \left(\frac{1}{3} + \frac{\eta^2}{v_0^2} k^2 \right) \frac{v_0}{\eta k} \operatorname{arctg} \frac{v_0}{\eta k} \right] \\ &= \frac{ar}{(3|v_0|x)} \left\{ \left(\frac{3}{4} (x_1^2 - x_2^2) + [E_1(x_2) + \log x_2] \frac{x_2^2}{2} - [E_1(x_1) + \log x_1] \frac{x_1^2}{2} \right. \right. \\ &\quad \left. \left. - \frac{\exp(-x_1) - \exp(-x_2)}{2} + \exp(-x_1) \frac{x_1}{2} - \exp(-x_2) \frac{x_2}{2} \right) x'^{-2} \right. \\ &\quad \left. + [E_1(x_2) + \log x_2 - E_1(x_1) - \log x_1] \left(\frac{1}{3} - \frac{4}{x'^2} \right) \right. \\ &\quad \left. + \left[\exp(-x_2) \left(\frac{1}{x_2} + \frac{1}{x_2^2} \right) - \exp(-x_1) \left(\frac{1}{x_1} + \frac{1}{x_1^2} \right) + \frac{1}{x_1^2} - \frac{1}{x_2^2} \right] \left(\frac{3}{x'^2} - \frac{4}{3} \right) \right. \\ &\quad \left. + \exp(-x_2) \left(\frac{1}{x_2} + \frac{3}{x_2^2} + \frac{6}{x_2^3} + \frac{6}{x_2^4} \right) - \exp(-x_1) \left(\frac{1}{x_1} + \frac{3}{x_1^2} + \frac{6}{x_1^3} + \frac{6}{x_1^4} \right) + \frac{6}{x_1^4} - \frac{6}{x_2^4} \right. \\ &\quad \left. + [x_2 - x_1 + x_1[E_1(x_1) + \log x_1] - x_2[E_1(x_2) + \log x_2]] \right. \\ &\quad \left. + \exp(-x_2) \left(1 - \frac{1}{x_2} + \frac{6}{x_2^2} + \frac{6}{x_2^3} \right) - \exp(-x_1) \left(1 - \frac{1}{x_1} + \frac{6}{x_1^2} + \frac{6}{x_1^3} \right) + \frac{4}{x_2} - \frac{6}{x_2^3} - \frac{4}{x_1} + \frac{6}{x_1^3} \right] x'^{-1} \Big\} \end{aligned}$$

where $R = v_0 r/\eta$ is the Reynolds number. For $x = r$ we have

$$A_3^0(r) = \frac{a}{v_0} \left[\frac{2}{3} R - \frac{1}{4} R^2 + O(R^3) \right]. \quad (25)$$

This determines the constant a by the requirement that the velocity $\mathbf{w}'(r)$ in the rest system of the sphere should be zero on the sphere; that means

$$\mathbf{w}(r) = \mathbf{w}'(r) - v_0 = -v_0 = (v_3, f)_v = A_3^0(r) \quad (26)$$

provided all other terms $\ell > 0$ in Eq. (22) contribute nothing, which will be investigated below. From Eqs. (26) and (25) we get

$$a = \frac{-v_0^2}{[\frac{2}{3}R - \frac{1}{4}R^2 + O(R^3)]} = -\frac{3}{2} \frac{v_0^2}{R} [1 + \frac{3}{8}R + O(R^2)]. \quad (27)$$

Inserting this value of a into Eq. (20) we obtain the drag

$$P = -4\pi ar^2 = 6\pi v_0 r \eta (1 + \frac{3}{8}R + \dots). \quad (28)$$

This is the well-known result of Oseen. Using the first approximation

$$a = -\frac{3}{2} \frac{v_0^2}{R} \quad (29)$$

in Eq. (24), in the lowest order of R we have

$$A_3^0(x) = -\frac{v_0 r}{x}$$

which is exactly what one gets from Stokes' solution.

Next, we turn to the term with $\ell = 2$ in (22). By the same techniques we obtain

with $x' = v_0 x / \eta$ and the quantities defined in Eq. (23). If this is expanded for $v_0 / \eta \rightarrow 0$, we get

$$A_3^2(x) = \frac{ar}{3v_0 x} \left\{ R \left(1 - \frac{r^2}{x^2} \right) + O \left[\left(\frac{v_0}{\eta} \right)^3 \right] \right\}.$$

The contribution proportional to $(v_0 / \eta)^2$ vanishes. That means that the boundary condition is satisfied by A_3^2 up to $O(R^3)$ or up to $O(R^2)$ if the factor a is taken into account. For this reason the quadratic term in Eqs. (25) and (27) can be taken seriously. Inserting the value (29) for a we again arrive at Stokes' solution

$$A_3^2(x) = \frac{\frac{1}{2} v_0 r}{x} \left\{ \frac{r^2}{x^2} - 1 \right\}.$$

Now, we calculate the terms $A_1(x)$ and $A_2(x)$ proportional to v_1 and v_2 , respectively, in the total solution (21).

$$\begin{aligned} A_1(x) &= -\frac{2ar}{(2\pi)^2} \int \frac{\sin^2 \alpha \cos \alpha \cos \beta}{iv_0 \cos \alpha + \eta k} \sin kr \\ &\quad \cdot \exp [ikx(\sin \alpha \sin \vartheta \cos(\beta - \varphi) \\ &\quad + \cos \alpha \cos \vartheta)] d\beta d\alpha dk \\ &= -i \frac{2ar}{2\pi} \cos \varphi \int \frac{\sin^2 \alpha \cos \alpha}{iv_0 \cos \alpha + \eta k} \sin kr \\ &\quad \cdot J_1(kx \sin \alpha \sin \vartheta) \exp(ikx \cos \alpha \cos \vartheta) d\alpha dk. \end{aligned}$$

Here, we expand in terms of Gegenbauer polynomials⁶ $C_\ell^{3/2}$

$$\begin{aligned} J_1(kx \sin \alpha \sin \vartheta) \exp(ikx \cos \alpha \cos \vartheta) \\ = (2\pi/kx)^{1/2} \sin \vartheta \sin \alpha \sum_{\ell=0}^{\infty} i^\ell (\ell + \frac{3}{2}) [(\ell+1)(\ell+2)]^{-1} \\ \cdot J_{\ell+3/2}(kx) C_\ell^{3/2}(\cos \alpha) C_\ell^{3/2}(\cos \vartheta), \end{aligned}$$

then we have

$$\begin{aligned} A_1(\mathbf{x}) &= \cos \varphi \sin \vartheta \sum_{\ell} A_1^\ell(x) C_\ell^{3/2}(\cos \vartheta), \\ A_2(\mathbf{x}) &= \sin \varphi \sin \vartheta \sum_{\ell} A_1^\ell(x) C_\ell^{3/2}(\cos \vartheta). \end{aligned}$$

Only the term with $\ell = 1$ contributes to the drag. In the same way as above we obtain

$$\begin{aligned} A_1^1(x) &= \frac{\frac{5}{4} ar}{|v_0| x} \\ &\quad \cdot \left\{ \left[E_1(x_2) + \log x_2 - E_1(x_1) - \log x_1 + \frac{1}{x_2^2} - \frac{1}{x_1^2} \right. \right. \\ &\quad + \exp(-x_1) \left(\frac{1}{x_1} + \frac{1}{x_1^2} \right) - \exp(-x_2) \left(\frac{1}{x_2} + \frac{1}{x_2^2} \right) \left. \right] \frac{3}{x'^2} \\ &\quad + \exp(-x_1) \left(\frac{2}{x_1^2} + \frac{6}{x_1^3} + \frac{6}{x_1^4} \right) \end{aligned}$$

$$\begin{aligned} &- \exp(-x_2) \left(\frac{2}{x_2^2} + \frac{6}{x_2^3} + \frac{6}{x_2^4} \right) + \frac{6}{x_2^4} - \frac{1}{x_2^2} - \frac{6}{x_1^4} + \frac{1}{x_1^2} \\ &+ \left[\frac{1}{x_1} - \frac{2}{x_1^3} - \frac{1}{x_2} + \frac{2}{x_2^3} + \exp(-x_1) \left(\frac{2}{x_1^2} + \frac{2}{x_1^3} \right) \right. \\ &\quad \left. - \exp(-x_2) \left(\frac{2}{x_2^2} + \frac{2}{x_2^3} \right) \right] \frac{3}{x'} \}. \end{aligned}$$

Expanding this for $v_0 / \eta \rightarrow 0$ we arrive at

$$A_1^1(x) = \frac{ar}{6v_0 x} \left\{ R \left(1 - \frac{r^2}{x^2} \right) + O \left[\left(\frac{v_0}{\eta} \right)^3 \right] \right\}.$$

The term proportional to $(v_0 / \eta)^2$ vanishes again. With the value (29) we get Stokes' result

$$A_1^1(x) = \frac{v_0 r}{4x} \left(\frac{r^2}{x^2} - 1 \right).$$

Summing up, the first mode contribution is

$$\begin{aligned} f^1(\mathbf{x}, \mathbf{v}) &= v_1 [A_1^0(x) + A_1^1(x) 3 \cos \vartheta + \dots] \sin \vartheta \cos \varphi \\ &\quad + v_2 [A_1^0(x) + A_1^1(x) 3 \cos \vartheta + \dots] \sin \vartheta \sin \varphi \\ &\quad + v_3 [A_3^0(x) + A_3^1(x) \cos \vartheta + A_3^2(x) \\ &\quad \cdot (\frac{3}{2} \cos^2 \vartheta - \frac{1}{2}) + \dots]. \end{aligned}$$

The terms with positive parity A_1^1 , A_3^0 , A_3^2 satisfy the usual boundary condition (26) up to $O(R^2)$. The situation is less satisfactory for the terms with negative parity. For instance for A_3^1 we obtain

$$\begin{aligned} A_3^1(x) &= -\frac{3ar}{2v_0 x} \\ &\quad \cdot \left\{ \left[E_1(x_1) + \log x_1 - E_1(x_2) - \log x_2 \right. \right. \\ &\quad + \left(\frac{1}{x_2} + \frac{1}{x_2^2} \right) \exp(-x_2) - \left(\frac{1}{x_1} + \frac{1}{x_1^2} \right) \\ &\quad \cdot \exp(-x_1) + \frac{1}{x_1^2} - \frac{1}{x_2^2} \left. \right] x'^{-1} \\ &\quad + \left(\frac{2}{x_2^2} + \frac{2}{x_2^3} \right) \exp(-x_2) - \left(\frac{2}{x_1^2} + \frac{2}{x_1^3} \right) \\ &\quad \cdot \exp(-x_1) + \frac{1}{x_2} - \frac{2}{x_2^3} - \frac{1}{x_1} + \frac{2}{x_1^3} \left. \right\} \end{aligned}$$

and after expanding

$$A_3^1(x) = \frac{3}{10} R v_0 \frac{r}{x} \left(\frac{1}{3} \frac{r}{x} - \frac{x}{r} \right)$$

which vanishes at only $O(R)$ on the sphere. This situation is the same as in Oseen's treatment of the problem. Since this term does not contribute to the drag, the formula (28) can be maintained, but the problem of the boundary condition on the sphere

must be more closely investigated. We return to this point in the last section.

Finally, let us consider the contributions f^4 and f^5 from the fourth and fifth modes. We have

$$(g_{4,5}, h)_v = a \cos \alpha [\pm 2^{-1/2} - ik\eta_1(\frac{2}{15})^{1/2}] \delta(x - r)$$

and

$$f_{4,5}(\mathbf{k}) = a \cos \alpha [\pm 2^{-1/2} - ik\eta_1(\frac{2}{15})^{1/2}] 4\pi r^2 \frac{\sin kr}{kr}$$

for the Fourier coefficient (8). This gives

$$f^{4,5}(\mathbf{x}, \mathbf{v})$$

$$= -\frac{4\pi ar^2}{(2\pi)^3} \int d^3k \frac{\pm 2^{-1/2} - ik\eta_2}{\mp ick - ikv_0 \cos \alpha - \Gamma k^2} \cos \alpha \cdot \frac{\sin kr}{kr} [(\frac{2}{15})^{1/2} u \pm ik\lambda_2 u + \dots] \exp(i\mathbf{k}\mathbf{x}),$$

where

$$\eta_2 = (\frac{2}{15})^{1/2} \eta_1, \quad \lambda_2 = 2^{-1/2} \lambda_1.$$

We pick up the terms with $u = \frac{1}{2}v^2$ which give the pressure. The contribution $B(\mathbf{x})$ from both modes is

$$B(\mathbf{x}) = \frac{2ar}{(2\pi)^2} \int \frac{(2^{-1/2} - ik\eta_2)[(\frac{2}{15})^{1/2} + ik\lambda_2]}{ic + iv_0 \cos \alpha + \Gamma k} + \frac{(-2^{-1/2} - ik\eta_2)[(\frac{2}{15})^{1/2} - ik\lambda_2]}{-ic + iv_0 \cos \alpha + \Gamma k} \sin \alpha \cos \alpha \sin kr \cdot \exp[ikx(\sin \alpha \sin \vartheta \cos(\beta - \varphi) + \cos \alpha \cos \vartheta)] dk d\alpha d\beta.$$

Here, the numerators must be calculated only up to $O(k)$, because the normal solutions (18) are only correct up to this order in the Navier-Stokes approximation. After integration with respect to β it is convenient to expand in terms of Legendre polynomials again

$$B(\mathbf{x}) = \sum_{\ell=0}^{\infty} B_{\ell}(x) P_{\ell}(\cos \vartheta).$$

The term which contributes to the drag is that with $\ell = 1$. For this we obtain

$$B_1(x) = \frac{3ar}{x^2} \left[\frac{2rc15^{-1/2}(\frac{1}{3}c^2 - v_0^2)}{(c^2 - v_0^2)^2} + \exp\left(-\frac{c}{\Gamma}(x-r)\right)[\dots] + \exp\left(-\frac{c}{\Gamma}(x+r)\right)[\dots] \right]. \quad (30)$$

The two exponentials which for brevity we have not written down in detail, give a boundary layer of thickness c/Γ which is of the order of magnitude of the mean free path. We expand the first term in

Eq. (30) for $v_0/c \rightarrow 0$. Let us remark that the expansion parameter for the pressure is the Mach number and not the Reynolds number as previously for the velocity and the stress tensor. Apart from the boundary layer we get

$$B_1(x) = \frac{2ar^2}{x^2} + O\left[\left(\frac{v_0}{c}\right)^2\right] = -\frac{3v_0\eta r}{x^2} + O\left[\left(\frac{v_0}{c}\right)^2\right].$$

Hence,

$$(f^4 + f^5)(\mathbf{x}, \mathbf{v}) = -\frac{3v_0\eta}{x^2} \cos \vartheta u$$

which gives the pressure

$$p(\mathbf{x}) = \frac{1}{3}(v^2, f)_v = \frac{5}{2}(f^4 + f^5) = -\frac{3v_0\eta r \cos \vartheta}{x^2}.$$

Again this is Stokes' result. The deviation from it in the general result (30) is $O[v_0/c]^2$. The calculation of the temperature and the density is also rather straightforward.

IV. STOKES' PROBLEM FOR THE CYLINDER

We take cylindrical coordinates

$$\mathbf{x} = (x_1 \cos \varphi, x_1 \sin \varphi, x_3),$$

$$\mathbf{k} = (k_1 \cos \beta, k_1 \sin \beta, k_3).$$

As a source function we take

$$h(\mathbf{x}, \mathbf{v}) = av_1 \delta(x_1 - r) \quad (31)$$

which produces the drag

$$P = -2\pi ar. \quad (32)$$

With the expression (31) all Fourier coefficients (8) are proportional to

$$\int \delta(x_1 - r) \exp(-i\mathbf{k}\mathbf{x}) d^3x = 2\pi r J_0(k_1 r) \delta(k_3).$$

Therefore, we must have $k_3 = 0$, which means that the x_3 dependence becomes trivial, and we only have to calculate in two dimensions. This simplifies the normal solutions to

$$\begin{aligned} \mathbf{x} &= x(\cos \varphi, \sin \varphi), & \mathbf{k} &= k(\cos \beta, \sin \beta), \\ \mathbf{e}^3 &= (\cos \beta, \sin \beta), & \mathbf{e}^1 &= (\sin \beta, -\cos \beta), \end{aligned}$$

$$\begin{aligned} g_1(\mathbf{k}, \mathbf{v}) &= v_1 \sin \beta - v_2 \cos \beta \\ &\quad - ik[T_{11} \cos \beta \sin \beta - T_{12} \cos^2 \beta \\ &\quad + T_{21} \sin^2 \beta - T_{22} \sin \beta \cos \beta], \end{aligned} \quad (33)$$

$$g_2(\mathbf{k}, \mathbf{v}) = v_3 - ik[T_{13} \cos \beta + T_{23} \sin \beta],$$

$$\begin{aligned} g_3(\mathbf{k}, \mathbf{v}) &= (\frac{2}{5})^{1/2} [u - \frac{5}{2} + ik\frac{2}{5}\lambda(v_1 \cos \beta + v_2 \sin \beta) \\ &\quad - i\mathbf{S}\mathbf{k}], \end{aligned}$$

$$g_{4.5} = (\frac{2}{15})^{1/2} [u - ik\eta_1(v_1 \cos \beta + v_2 \sin \beta) - i\mathbf{S} \cdot \mathbf{k}] \pm 2^{-1/2} [v_1 \cos \beta + v_2 \sin \beta + ik(\frac{2}{5}\lambda + \lambda_1 u) - ik \dots].$$

Again the first mode gives the leading contribution to the velocity. We get

$$(g_1, h)_v = a \sin \beta \delta(x - r),$$

$$f^1(\mathbf{k}) = 2\pi a r \sin \beta J_0(kr),$$

$$f^1(\mathbf{x}, \mathbf{v}) = -\frac{2\pi a r}{(2\pi)^2} \int \frac{\sin \beta}{-ikv_0 \cos \beta - \eta k^2} [v_1 \sin \beta - v_2 \cos \beta - ik \dots] J_0(kr) \exp(i\mathbf{kx}) d^2k.$$

Let us discuss the behavior for $x \rightarrow \infty$ in this Fourier integral. The term being most singular at $\mathbf{k} = 0$ is the term proportional to v_1 , say $A_1(\mathbf{x})$. We can easily estimate that

$$\frac{\sin^2 \beta}{-ikv_0 \cos \beta - \eta k^2} J_0(kr) \in L^p(\mathbf{R}^2), \quad 1 < p < \frac{3}{2}.$$

Applying the Hausdorff–Yang theorem it follows that

$$A_1(\mathbf{x}) \in L^{3+\epsilon}(\mathbf{R}^2), \quad \epsilon > 0.$$

Hence, Stokes' paradox does not occur.

Now, we calculate $A_1(\mathbf{x})$ in detail

$$A_1(\mathbf{x}) = \frac{ar}{2\pi} \int \frac{\sin^2 \beta}{iv_0 \cos \beta + \eta k} J_0(kr) \cdot \exp[ikx \cos(\beta - \varphi)] d\beta dk.$$

Analogous to the three-dimensional case the exponential is expanded into a Fourier series

$$\begin{aligned} \exp[ikx \cos(\beta - \varphi)] &= J_0(kx) + 2 \sum_{n=1}^{\infty} (-1)^n [\cos 2n\beta \cos 2n\varphi \\ &+ \sin 2n\beta \sin 2n\varphi] J_{2n}(kx) \\ &+ 2i \sum_{n=0}^{\infty} (-1)^n [\cos(2n+1)\beta \cos(2n+1)\varphi \\ &+ \sin(2n+1)\beta \sin(2n+1)\varphi] J_{2n+1}(kx). \end{aligned}$$

Then, we have

$$A_1(\mathbf{x}) = \sum_{\ell} [A_1^{\ell}(x) \cos \ell\varphi + \tilde{A}_1^{\ell}(x) \sin \ell\varphi]. \quad (34)$$

The terms which contribute to the drag applying the Navier–Stokes relations, are A_1^0 and A_2^0 . For A_1^0 we get

TABLE I. Drag coefficient $D(R) = P/(\rho v_0^2 r)$ of a circular cylinder at low Reynolds numbers $R = v_0 r / \eta$ (r is the radius of the cylinder).

R	$D(R)$	R	$D(R)$
0.1	34.75	1.1	8.27
0.2	21.43	1.2	7.96
0.3	16.50	1.3	7.71
0.4	13.87	1.4	7.48
0.5	12.21	1.5	7.28
0.6	11.06	1.6	7.10
0.7	10.22	1.7	6.95
0.8	9.56	1.8	6.80
0.9	9.04	1.9	6.67
1.0	8.62	2.0	6.55

$$\begin{aligned} A_1^0(x) &= \frac{ar}{2\pi} \int \frac{\sin^2 \beta}{iv_0 \cos \beta + \eta k} J_0(kr) J_0(kx) d\beta dk \\ &= \frac{ar}{2\pi} \frac{2\pi\eta}{v_0^2} \cdot \int_0^{\infty} dk \left[-1 + \left(1 + \frac{v_0^2}{\eta^2 k^2} \right)^{1/2} \right] k J_0(kr) J_0(kx). \end{aligned}$$

Introducing dimensionless variables

$$k' = \frac{k\eta}{v_0}, \quad x' = \frac{xv_0}{\eta}, \quad R = \frac{rv_0}{\eta} \quad (35)$$

we have

$$A_1^0(x) = \frac{aR}{v_0} \int_0^{\infty} dk' [(k'^2 + 1)^{1/2} - k'] J_0(k'R) J_0(k'x'),$$

the integral could not be carried out analytically.

The condition of velocity 0 at the cylinder (in the rest frame of the cylinder) now requires

$$A_1^0(r) = -v_0$$

which for the constant a gives

$$a = -v_0^2 \left(R \int_0^{\infty} dk [(k^2 + 1)^{1/2} - k] J_0^2(kR) \right)^{-1}. \quad (36)$$

Inserting this value in Eq. (32) we obtain the following drag formula:

$$\begin{aligned} P &= -2\pi a r = v_0^2 r D(R), \\ D(R) &= 2\pi \left(R \int_0^{\infty} dk [(k^2 + 1)^{1/2} - k] J_0^2(kR) \right)^{-1}. \end{aligned} \quad (37)$$

From this result (37) we have evaluated the drag coefficient $D(R)$ numerically. The results are given in Table I and are plotted in Fig. 1. The classical result of Lamb, the result of Kaplun,⁷ and experimental values of Tritton⁵ are shown for comparison. For small Reynolds numbers R our results agree with those of Lamb, for higher Reynolds numbers $R > 0.5$ they are in better agreement with the experimental values than Lamb's and Kaplun's

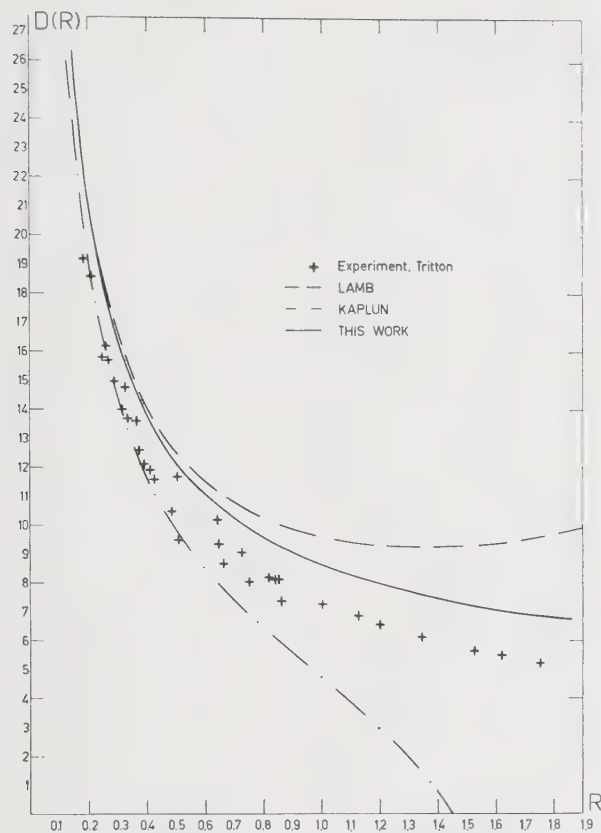


FIG. 1. Drag coefficient $D(R)$ of a circular cylinder at low Reynolds numbers R .

results. But there is a systematic deviation of the experimental points to lower values, even for small R where there is no doubt of the validity of the theory.

Now, we turn to the $\ell = 2$ terms in Eq. (34). We get

$$\tilde{A}_1^2 = 0$$

and

$$A_1^2(x) = -\frac{4aR}{v_0} \int_0^\infty dk' k' \left[1 + k'^2 - \left(k'^2 + \frac{1}{2} \right) \cdot \left(1 + \frac{1}{k'^2} \right)^{1/2} \right] J_0(k'R) J_2(k'x').$$

This integral should vanish for $x = r$ and $R \rightarrow 0$. In this limit the main contribution to the integral comes from large values of k' . Then, expanding the square root we have

$$\int_0^\infty dk' J_0(k'R) J_2(k'R) / k' = 0 \quad (38)$$

which indeed vanishes. Therefore, the usual boundary condition $[\mathbf{w}'(r) = 0]$ at the cylinder is fulfilled for small R . That is true all the more because $A_1^0(r) \rightarrow \infty$ for $R \rightarrow 0$.

Concerning the term $A_2(\mathbf{x})$ proportional to v_2 in Eq. (33) we proceed in the same way. We obtain a Fourier series as in Eq. (34) and the interesting term contributing to the drag is now

$$\tilde{A}_2^2(x) = \frac{4aR}{v_0} \int_0^\infty dk' k' \left[\frac{1}{2} + k'^2 - k'(1 + k'^2)^{1/2} \right] \cdot J_0(k'R) J_2(k'x').$$

If the square root is expanded in the limit $R \rightarrow 0$, we get the integral (38) again. Hence, this term, too, satisfies the usual boundary condition for small Reynolds numbers.

Finally let us consider the contributions to the pressure from the fourth and fifth modes. The result is

$$(f^4 + f^5)(\mathbf{x}, \mathbf{v}) = -\frac{2}{5} \cos \varphi \frac{ar}{x} u + \text{boundary layer}$$

which gives a pressure

$$\begin{aligned} p(\mathbf{x}) &= \frac{1}{3}(v^2, f)_v = -a \cos \varphi \frac{r}{x} + \text{boundary layer} \\ &= -\frac{v_0^2 r \cos \varphi D(R)}{2\pi x} + \text{boundary layer.} \end{aligned}$$

The main term in this last expression for the pressure in terms of the drag coefficients $D(R)$ is precisely what one obtains also from Lamb's hydrodynamical solution of the problem.

V. DISCUSSION

In the preceding calculations we obtained results in the kinetic treatment similar to those known from hydrodynamical calculations using the linearized Oseen equations. This must be so because we have used the linearized definitions of the hydrodynamical quantities

$$n(\mathbf{x}) = (1, f)_v, \quad \mathbf{w}(\mathbf{x}) = (\mathbf{v}, f)_v$$

and the corresponding hydrodynamical equations are just the Oseen equations. However we also could take the exact (nonlinear) definitions

$$(1, f)_v = n(\mathbf{x}), \quad (\mathbf{v}, f)_v = [1 + n(\mathbf{x})]\mathbf{w}(\mathbf{x}).$$

Then, nonlinear effects would be taken into account. It seems to the author that this refinement is less important than another one, i.e., modification of the boundary condition.

The boundary condition on the surface of the body is the problem. Close to the boundary there exists the kinetic boundary layer. Within this layer hydrodynamics breaks down and so does the normal solution method. The reason for this is

that the microscopic relaxation becomes very important.⁴ Unfortunately, the latter is not tractable at present (apart from a few special models). However, the complicated problem of the microscopic relaxation can be completely avoided if the boundary condition at the body is known, because then the solution outside the layer can be calculated exactly by the normal solution method. The problem of finding the boundary condition seems to be simpler than the full treatment of the microscopic relaxation.

The boundary condition chosen here is only a first approximation because it neglects the boundary layer completely. To understand this we should remark that simultaneously with the external Stokes' problem we have solved an internal problem of a fluid moving inside a sphere (one has only to compute the Fourier integrals for $|\mathbf{x}| < r$. With the boundary condition $\mathbf{w} = 0$ on the surface this internal problem is the trivial one of a fluid at rest with respect to the sphere. Therefore, the stress tensor vanishes inside the sphere. Consequently,

the momentum produced on the surface is entirely transferred to the outer fluid. Since we have distributed the sources of momentum uniformly, the force per unit area on the sphere in the direction of motion must be constant over the surface. This is exactly what one gets if the hydrodynamical (Stokes') solution is extrapolated to the boundary. At higher Reynolds numbers this is no longer possible and the boundary condition must be modified.

ACKNOWLEDGMENT

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